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MATH305: THEORY OF RINGS AND FIELDS 2023/2024 EXAMINATION

ATTEMPT ANY FOUR QUESTIONS TIME: 2 HOURS

1. a) Distinguish between an integral domain and a division ring.
   b) Prove that a commutative ring R with unity whose only ideals are $\{0\}$ and R itself is necessarily a field.
2. a) When is a ring said to be: (i) Euclidean (ii) Principal ideal domain.
   b) Show that in a Euclidean ring R, the greatest common divisor of any two elements a, b exists and is of the form $\lambda a + \mu b$, with $\lambda \in R$.
3. a) Define and exemplify the concept of maximal ideal in a ring.
   b) Prove that if R is a commutative ring with unity, then an ideal M of R is maximal if and only if $R/_M$ is a field.
4. a) Define an ideal of a ring and prove that the kernel of a ring homomorphism $\theta: R \rightarrow R'$ is an ideal of R.
   b) Define a principal ideal domain and show that every Euclidean ring is a principal ideal domain.
5. a) Let R be the ring of real valued continuous functions on the closed unit interval $[0,1]$. Prove that the subset $M = \{f(x) \in R : f(^2/_3) = 0\}$ is a maximal ideal of R.
   b) If $f(x), g(x)$ are non-zero polynomials in $F[x]$, prove that $\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$.
6. a) If $f(x), g(x) \in F[x]$ with $g(x) \neq 0$, prove that there exist $r(x), t(x) \in F[x]$ such that $f(x) = g(x)t(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.
   b) Show that a primitive polynomial is irreducible over the integers if and only if it is irreducible over the rationals.

MATH 305 ANSWERS

1a. Distinguish between an integral domain and a ~~com~~ division ring

Ans

An integral domain, is a commutative ring R in which there are no zero divisors. i.e $\forall a, b \in R$, $a.b = 0 \Rightarrow a = 0$ or $b = 0$

While a division ring R is a ring which is not necessarily commutative but every element in R, has a multiplicative Inverse.

NOTE: A commutative division ring is a field.

1b. Prove that in a commutative ring R with Unity whose only ideals are $\{0\}$ and R itself is necessarily a field.

PROOF

Let $a \neq 0 \in R$ and consider the set $Ra := \{ra \mid r \in R\}$

Then Ra is an ideal of R, for if $s \in R$ and $ra \in Ra$, then $s(ra) = (sr)a \in Ra$

Also, if $r_1 a, r_2 a \in Ra$, then

$$(r_1 a - r_2 a) = (r_1 - r_2)a \in Ra.$$

Now since R has a unity 1,

$a = 1.a \in Ra$ and so $Ra \neq \{0\}$

Therefore by our assumption $Ra = R$.

Since $1 \in R = Ra$, there is an element say $b \in R$ such that $ba = 1$ □.

(2) When is a ring said to be (i) Euclidean (ii) principal ideal domain.

2b. Show that in a Euclidean ring R, the greatest common divisor of any two elements a, b exists and is of the form $\lambda a + \mu b$.

Proof

Let $A = \{ta + sb \mid t, s \in R\}$.

Then A is an ideal of R, for if $t_1 a + s_1 b,\ t_2 a + s_2 b \in A$, then

$(t_1 a + s_1 b) - (t_2 a + s_2 b) = (t_1 - t_2)a + (s_1 - s_2)b \in A$.

So that A is a subgroup under addition.

Also, if $ta + sb \in A$, and $r \in R$, then

$$(ta + sb)r = (tr)a + (sr)b \in A.$$

Since R is a principal ideal domain (PID), $A = \langle d \rangle$ for some $d \in A$

$a = 1 \cdot a + 0 \cdot b \in A$ and

$b = 0 \cdot a + 1 \cdot b \in A$.

So that $d \mid a$ and $d \mid b$

If $c \in R$ is such that $c \mid a$, and $c \mid b$, then $a = t_1 c$ and $b = s_1 c$ for some $t_1, s_1 \in R$. But as $d \in A$, there are λ and $\mu \in R$ such that

$d = \lambda a + \mu b$
$\quad = \lambda t_1 c + \mu s_1 c = (\lambda t_1 + \mu s_1)c$ implying that $c \mid d$. Thus

$d = \gcd(a, b)$ and $d = \lambda a + \mu b$ $\square$.

3a) Define and exemplify the concept of maximal ideal in a ring

Definition (maximal ideal). Let R be a ring and M an ideal of R. M is called maximal ideal if, for all ideal I of R, $M \subseteq I \subseteq R \Rightarrow M = I$ or $I = R$

Example.

Let $R := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ be the ring of real-valued continuous functions (maps) on the closed bounded interval $[0,1]$. Let $M = \{f \in R \mid f(1/2) = 0\}$. Then M defined is a maximal ideal of R.

3b. Prove that if R is a commutative ring with Unity, then an ideal M of R is maximal If and only if R/M is a field.

Proof

Suppose R/M is a field, then the only ideals of R/M are $\{0\}$ and R/M itself. Since $\pi: R \to R/M$ given by $\pi(r) = r + M$ is a homomorphism of R onto R/M with Kernel M, it follows by correspondence-theorem that there is a one-one correspondence between the ideals of R/M and the ideals of R containing M. Thus, since by our assumption R/M has only two ideals $\{0\}$ and R/M, it follows that there are only two ideals of R containing M namely M and R. Hence, M is maximal in R.

Conversely, suppose M is maximal ideal of R. Then, again by the correspondence theorem, there is a one-one correspondence between ideals of R ~~and~~ containing M and ideals of R/M. Since by our assumption M is maximal, then the only ideals of R containing M are M and R. Thus R/M has only two ideals namely $\{0\}$ and R/M. Since R is commutative with Unity, R/M too is commutative with Unity and thus a field as required □.

4a. Define an ideal of a ring and prove that the kernel of a ring homomorphism $\theta: R \to R'$ is an ideal of R.

Definition (ideal of a ring R):

Let R be a ring. A subgroup A of R under addition is said to be

(i) ~~A~~ Left ideal, if for all $a \in A$ and $r \in R$, $ra \in A$.

(ii) ~~A~~ right ideal if for all $a \in A$ and $r \in R$ $ar \in A$

(iii) An ideal if A is both left and right ideal.

4a(ii) Kernel of the ring homomorphism $\theta: R \to R'$ is an ideal of R.

PROOF

Recall, that if θ is a homomorphism, then $Ker\theta := \{r \in R \mid \theta(r) = 0 \in R'\}$.

first we show that the $Ker\theta$ is a subgroup under addition.

Now, Let x, and $y \in Ker\theta$, Then $\theta(x) = \theta(y) = 0$

$\Rightarrow \theta(x) - \theta(y) = 0$

$\Rightarrow \theta(x-y) = 0$ since θ is a homomorphism.

$\Rightarrow x-y \in Ker\theta$ and so $Ker\theta$ is a subgroup of R under addition.

Suppose that $x \in Ker\theta$, and $r \in R$,

then $\theta(x) = 0 = \theta(0)$

thus $x \cdot r = r \cdot x = 0 \in Ker\theta$. $\square$.

4b. Define principal ideal domain and show that every Euclidean ring is a principal ideal domain.

~~Pr~~ Definition

Principal ideal Domain (PID):

An ideal I of R is said to be principal ideal domain, if $\exists a \in R \ni I = \langle a \rangle$

Every Euclidean ring Is a PID

Let R be a Euclidean ring and A be an ideal of R. If $A = \{0\}$, then A is principal. Suppose $A \neq \{0\}$ and let $a \neq 0$ in A be such that $d(a)$ is minimal in A. {i.e for any $b \in A$ $d(a) < d(b)$}

Let $b \neq 0$ be in A. Then, since R is a Euclidean ring, $\exists$ $t, r \in R$ such that $b = at + r$ where $r = 0$ or $d(r) < d(a)$. since $d(a)$ is minimial, we must have $r = 0$ and $b = at$. Thus $A = \{at \mid a \in R\}$ is a principal ideal.

It remains to show that R contains identity element. Since R is an ideal of itself, it means that there is an element $u \in R$ such that $R = \{ur \mid r \in R\}$.

But then u being in R, is a multiple of itself, that is there is $r_0 \in R$ such that $ur_0 = u$. Now if $b \in R$, then $b = ur$ for some $r \in R$ and

$br_0 = urr_0 = ur_0r = ur = b$. Thus r_0 is the identity element of R.

Hence, R is a PID $\square$.